LEARNING THEORY

The Advantage of Abstract Examples in Learning Math

Undergraduate students may benefit more from learning mathematics through a single abstract, symbolic representation than from learning multiple concrete examples.

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Abstract knowledge, such as mathematical knowledge, is often difficult to acquire and even more difficult to apply to novel situations (*1–3*). It is widely believed that a successful approach to this challenge is to present the learner with multiple concrete and highly familiar examples of the to-be-learned concept. For instance, a mathematics instructor teaching simple probability theory may present probabilities by randomly choosing a red marble from a bag containing red and blue marbles and by rolling a six-sided die. These concrete, familiar examples instantiate the concept of probability and may facilitate learning by connecting the learner's existing knowledge with new, to-be-learned knowledge. Alternatively, the concept can be instantiated in a more abstract manner as the probability of choosing one of *n* things from a larger set of *m* things.

The belief in the effectiveness of multiple concrete instantiations is reasonable: A student who sees a variety of instantiations of a concept may be more likely to recognize a novel analogous situation and apply what was learned. Learning multiple instantiations of a concept may result in an abstract, schematic knowledge representation (*1*, *4*), which, in turn, promotes knowledge transfer, or application of the learned concept to novel situations (*1*, *5*). However, concrete information may compete for attention with deep to-be-learned structure (*6–8*). Specifically, transfer of conceptual knowledge is more likely to occur after learning a generic instantiation than after learning a concrete one (*7*).

Therefore, we ask: Is learning multiple concrete instantiations the most efficient route to promoting transfer of mathematical knowledge? Here, we tested a hypothesis that learning a single generic instantiation (that is, one that communicates minimal extraneous information) may result in better knowledge transfer than learning multiple concrete, contextualized instantiations.

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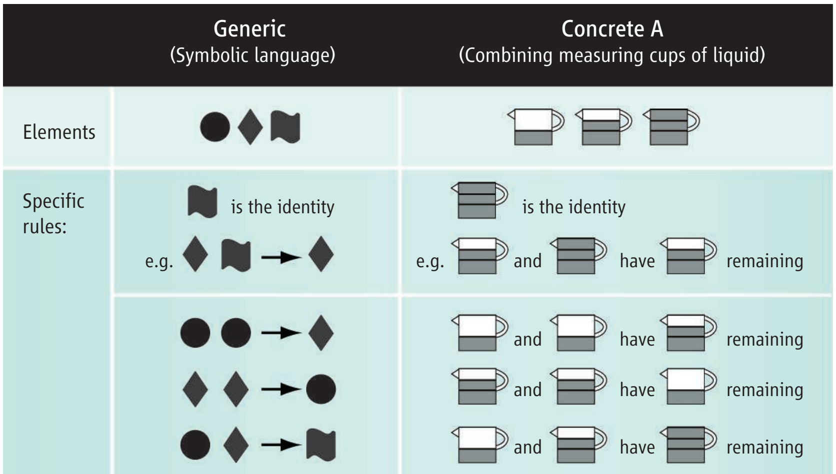
In experiment 1, undergraduate college students learned one or more instantiations of a simple mathematical concept. They were then presented with a transfer task that was a novel instantiation of the learned concept. The to-be-learned concept was that of a commutative mathematical group of order three. This concept is a set of three elements, or equivalence classes, and an operation with the associative and commutative properties, an identity element, and inverses for each element. This concept was chosen because it involves the most basic properties of the real number system, yet it is simple, novel to the study participants, and can be easily instantiated in different ways.



Generic and concrete instantiations of a mathematical group.

One instantiation used in this research was generic. This instantiation was described as a written language involving three symbols (see figure, above) in which combinations of two or more symbols yield a predictable resulting symbol. Statements were expressed as *symbol 1, symbol 2 → resulting symbol.* Three other instantiations (Concrete A, B, and C) were concrete, contextualized, and involved elements that might appear meaningful in the context. The Concrete A instantiation was shown in previous research to facilitate quick learning of the rules of the mathematical group (*6*). The elements were three images of measuring cups containing varying levels of liquid (see figure, below). Participants were told they needed to determine a remaining amount when different measuring cups of liquid are combined. Concrete B and C instantiations were constructed similarly, with story lines and elements that would assist learning. The same mathematical rules were presented in slices of pizza or tennis balls in a container, rather than portions of a measuring cup of liquid (*9*). Eighty study participants were assigned to one of four learning conditions: Generic 1, Concrete 1, Concrete 2, or Concrete 3, with participants learning one generic instantiation, one concrete instantiation, two concrete instantiations, or three concrete instantiations, respectively.

Training was equated across conditions; all participants were presented with the same rules and the same number of examples, questions with feedback, and test questions. After this learning phase, all participants were presented with the same transfer task, which was a novel concrete instantiation of the same group structure that was presented during learning. The transfer instantiation involved perceptually rich elements, as do many real-world instantiations of mathematics, and was described as a children's game involving three objects (*9*). In the game, children sequentially pointed to objects; and a child who was "it" pointed to a final object. If the child pointed to the correct final object, then he or she was the winner. The correct final object was specified by the rules of the game (rules of the mathematical group). Participants received no explicit training in the transfer domain. Instead, they were told that the rules of the game were like the rules of the system(s) they had just learned and that they could figure out these rules by using their newly acquired knowledge. After being asked to study

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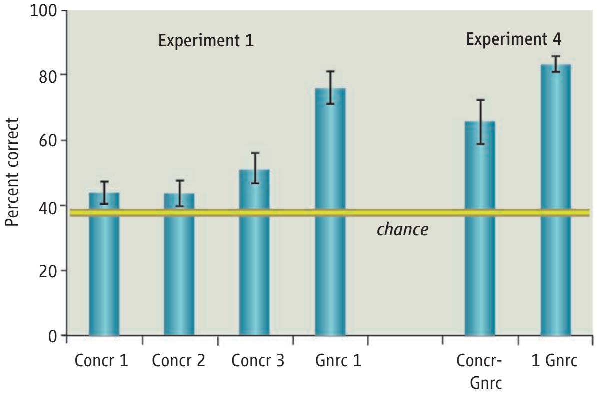

Transfer test scores across learning conditions (means ± SEM).

a series of examples, from which the rules could be deduced, they received a 24-question multiple-choice test isomorphic to the questions they answered during the learning phase.

In all conditions of this experiment (as well as the other experiments reported here), participants successfully learned the material with no differences in learning scores ($F_{3,68} < 1$) or learning times ($F_{3,68} < 1.5$). However, there were significant differences in transfer (see experiment 1, in the figure above). Participants in the Generic 1 condition performed markedly higher than participants in each of the three concrete conditions ($F_{3,68} = 11.9$, $P < 0.001$; post hoc Tukey's test, P values < 0.002). Furthermore, transfer in the Generic 1 condition was above chance ($t > 7$, $P < 0.005$), whereas transfer in the concrete conditions did not reliably exceed chance (t values < 1.7, P values > 0.35; $t = 2.8$, $P = 0.06$ for Concrete 3).

These results indicate that learning one, two, or three concrete instantiations resulted in little or no transfer, whereas learning one generic instantiation resulted in significant transfer. If transfer from multiple instantiations depends on whether the learner abstracts and aligns the common structure from the learned instantiations (*1*, *4*), then transfer failure suggests that participants may have been unable to recognize and align the underlying structure.

In two additional experiments, we assisted structural alignment. In experiment 2, 20 participants learned Concrete A and Concrete B instantiations and were given the alignment of analogous elements across the learning instantiations. To our surprise, this assistance yielded no improvement in transfer; scores were not above chance (means ± SD: 41% ± 16.7%, $t_{19} = 0.94$, $P > 0.35$). In experiment 3, we asked 20 participants after learning Concrete A and Concrete B instantiations to compare them, by matching analogous elements and writing any observed similarities. Explicit comparisons have been shown to facilitate transfer (*5*, *10*). All participants correctly matched elements, but the distribution of transfer scores was bimodal. Approximately 44% of our participants scored highly on the transfer test (95% ± 4.7%). However, the remaining participants did not do well (51% ± 11.6%). Therefore, the act of explicit comparison may help some, perhaps high-performing, learners transfer, but may not help others (*11*).

Overall, concrete and generic instantiations have different advantages. Concrete instantiations may be more engaging for the learner and may facilitate initial learning (*6*), but do not necessarily promote transfer. At the same time, generic instantiations can be learned and do promote transfer. On these grounds, one could argue that presenting a concrete instantiation and then a generic instantiation may be an optimal learning design for promoting transfer. One could also argue that the concrete instantiations used in experiments 1 to 3 are very similar to each other and that successful transfer might require instantiations that are more diverse.

We address these issues in experiment 4. Forty participants were assigned to one of two learning conditions: One-Generic (participants learned the generic instantiation) or Concrete-then-Generic (participants learned the Concrete A instantiation then the Generic instantiation). The results were that participants who learned only the generic instantiation outperformed those who learned both concrete and generic instantiations (see experiment 4 in the figure above; $t_{31} = 2.7$, $P < 0.02$).

Our findings suggest that giving college students multiple concrete examples may not be the most efficient means of promoting transfer of knowledge. Moreover, because the concept used in this research involved basic mathematical principles and test questions were both novel and complex, these findings could likely be generalized to other areas of mathematics. For example, solution strategies may be less likely to transfer from problems involving moving trains or changing water levels than from problems involving only variables and numbers. Instantiating an abstract concept in a concrete, contextualized manner appears to constrain that knowledge and to hinder the ability to recognize the same concept elsewhere; this, in turn, obstructs knowledge transfer. At the same time, learning a generic instantiation allows for transfer, which suggests that such an instantiation could result in a portable knowledge representation. Compared with concrete instantiations, generic instantiations present minimal extraneous information and hence represent mathematical concepts in a manner close to the abstract rules themselves.

Because the difficulty of transferring knowledge acquired from concrete instantiations may stem from extraneous information diverting attention from the relevant mathematical structure, concrete instantiations are also likely to hinder transfer for young learners who are less able than adults to control their attentional focus. We have evidence that 11-year-olds transferred successfully from a generic instantiation, but not from a concrete one (*12*).

If a goal of teaching mathematics is to produce knowledge that students can apply to multiple situations, then presenting mathematical concepts through generic instantiations, such as traditional symbolic notation, may be more effective than a series of "good examples." This is not to say that educational design should not incorporate contextualized examples. What we are suggesting is that grounding mathematics deeply in concrete contexts can potentially limit its applicability. Students might be better able to generalize mathematical concepts to various situations if the concepts have been introduced with the use of generic instantiations.

References and Notes

1. M. L. Gick, K. J. Holyoak, *Cogn. Psychol.* **15**, 1 (1983).
2. L. R. Novick, *J. Exp. Psychol. Learn. Mem. Cogn.* **14**, 510 (1988).
3. S. K. Reed, A. Dempster, M. Ettinger, *J. Exp. Psychol. Learn. Mem. Cogn.* **11**, 106 (1985).
4. L. R. Novick, K. J. Holyoak, *J. Exp. Psychol. Learn. Mem. Cogn.* **17**, 387 (1991).
5. R. Catrambone, K. J. Holyoak, *J. Exp. Psychol. Learn. Mem. Cogn.* **15**, 1147 (1989).
6. J. A. Kaminski, V. M. Sloutsky, A. F. Heckler, in *Proceedings of the 27th Annual Conference of the Cognitive Science Society*, B. Bara, L. Barsalou, M. Bucciarelli, Eds., Stresa, Italy, 21 to 23 July 2005 (Lawrence Erlbaum, Mahwah, NJ, 2005), pp. 1090–1095.
7. V. M. Sloutsky, J. A. Kaminski, A. F. Heckler, *Psychonom. Bull. Rev.* **12**, 508 (2005).
8. R. L. Goldstone, Y. Sakamoto, *Cogn. Psychol.* **46**, 414 (2003).
9. Materials and methods are available as supporting material on *Science* Online.
10. D. Gentner, J. Loewenstein, L. Thompson, *J. Educ. Psychol.* **95**, 393 (2003).
11. Learning scores differed between participants who transferred and those who did not (means ± SD: 93% ± 4.4% and 84% ± 12.9%, respectively), independent sample t test, $t_{16} = 1.97$, $P = 0.066$.
12. J. A. Kaminski, V. M. Sloutsky, A. F. Heckler, in *Proceedings of the 27th Annual Conference of the Cognitive Science Society*, R. Sun, N. Miyake, Eds., Vancouver, BC, 26 to 29 July 2006 (Lawrence Erlbaum, Mahwah, NJ, 2006), pp. 411–416.
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